RANK-ONE FLOWS OF TRANSFORMATIONS WITH INFINITE ERGODIC INDEX

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Abstract. A rank-one infinite measure preserving flow $T = (T_t)_{t\in\mathbb{R}}$ is constructed such that for each $t \neq 0$, the Cartesian powers of the transformation T_t are all ergodic.

0. Introduction

In 1963 Kakutani and Parry discovered an interesting phenomenon in the theory of infinite measure preserving maps. They showed that for each $p > 0$, there exists a transformation whose p-th Cartesian power is ergodic but $(p+1)$-th one is not [KP]. Since then a number of other examples of transformations with exotic (from the point of view of the classical "probability preserving" ergodic theory) weak mixing properties were constructed. See surveys [Da2] and [DaS3] for a detail discussion on that. In [Da1], [DaS1] these examples were extended to infinite measure preserving actions of discrete countable Abelian groups. Weak mixing properties of infinite measure preserving actions of *continuous* Abelian groups such as $\mathbb{R}$ and $\mathbb{R}^d$ were under consideration in [I–W]. In particular, a rank-one flow (i.e. $\mathbb{R}$-action) whose Cartesian square is ergodic was constructed there. A rank-one infinite measure preserving flow $T = (T_t)_{t\in\mathbb{R}}$ with infinite ergodic index (i.e. the Cartesian powers of T are all ergodic) appeared in a recent paper [DaSo]. It can be deduced easily from [DaSo] that there is a residual subset D_T of $\mathbb{R}$ such that for each $t \in D_T$, the transformation T_t has infinite ergodic index. However the following more subtle question by C. Silva remained open so far:

Is there a rank-one infinite measure preserving flow T with $D_T = \mathbb{R}\setminus\{0\}$?

Our purpose in this paper is to answer his question in the affirmative.

Theorem 0.1. *There is a rank-one infinite σ-finite measure preserving flow $T = (T_t)_{t\in\mathbb{R}}$ such that for each $t \neq 0$, the transformation T_t has infinite ergodic index.*

The main idea of the proof is different from those that were used in [I–W] and [DaSo]. It is based on a technique to *force* a dynamical property. Originating from [Ry1], such techniques were utilized in [Ry2], [DaR], etc. to obtain mixing, power weak mixing, etc. of some systems. In this paper the desired flow appears as a certain limit of a sequence of weakly mixing finite measure preserving flows. We construct this sequence in such a way to retain the property of infinite ergodic index

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in the limit. The construction is implemented in the language of (C,F)-actions (see [Da2]).

1. Preliminaries: rank-one actions and (C,F)-actions of $\mathbb{R}^d$

We first recall the definition of rank one. Let $S=(S_g)_{g\in\mathbb{R}^d}$ be a measure preserving action of $\mathbb{R}^d$ on a standard σ-finite measure space $(Y,\mathfrak{C},\nu)$.

Definition 1.1.

(i) A *Rokhlin tower or column* for S is a triple (A,f,F), where $A\in\mathfrak{C}$, F is a cube in $\mathbb{R}^d$ and $f:A\to F$ is a measurable mapping such that for any Borel subset $H\subset F$ and an element $g\in\mathbb{R}^d$ with $g+H\subset F$, one has $f^{-1}(g+H)=S_gf^{-1}(H)$.

(ii) S is said to be of *rank-one (by cubes)* if there exists a sequence of Rokhlin towers (A_n,f_n,F_n) such that the volume of F_n goes to infinity and for any subset $C\in\mathfrak{C}$ of finite measure, there is a sequence of Borel subsets $H_n\subset F_n$ such that
$$\lim_{n\to\infty}\nu(C\triangle f_n^{-1}(H_n))=0.$$

The (C,F)-construction of measure preserving actions for discrete countable groups was introduced in [dJ] and [Da1]. It was extended to the case of locally compact second countable Abelian groups in [DaS2]. (See also [Da2].) Here we outline it briefly for $\mathbb{R}^d$, $d\in\mathbb{N}$.

Given two subsets $E,F\subset\mathbb{R}^d$, by $E+F$ we mean their algebraic sum, i.e. $E+F=\{\,e+f\mid e\in E, f\in F\,\}$. The algebraic difference $E-F$ is defined in a similar way. If F is a singleton, say $F=\{f\}$, then we will write $E+f$ for $E+F$. Two subsets E and F of $\mathbb{R}^d$ are called *independent* if $(E-E)\cap(F-F)=\{0\}$, i.e. if $e+f=e'+f'$ for some $e,e'\in E$, $f,f'\in F$ then $e=e'$ and $f=f'$.

Fix $p\in\mathbb{N}$ and consider two sequences $(F_n)_{n=0}^\infty$ and $(C_n)_{n=1}^\infty$ of subsets in $\mathbb{R}^d$ such that F_n is a cube $[0,h_n)\times\cdots\times[0,h_n)$ (d times) for an $h_n\in\mathbb{R}$, $C_n\subset\mathbb{R}^d$ is a finite set, $\#C_n>1$,

$$F_n \text{ and } C_{n+1} \text{ are independent;} \tag{1-1}$$
$$F_n+C_{n+1}\subset F_{n+1}. \tag{1-2}$$

This means that F_n+C_{n+1} consists of $\#C_{n+1}$ mutually disjoint 'copies' F_n+c of F_n, $c\in C_{n+1}$, and all these copies are contained in F_{n+1}. We equip F_n with the measure $(\#C_1\cdots\#C_n)^{-1}(\lambda_{\mathbb{R}^d}\restriction F_n)$, where $\lambda_{\mathbb{R}^d}$ denotes Lebesgue measure on $\mathbb{R}^d$. Endow C_n with the equidistributed probability measure. Let $X_n:=F_n\times\prod_{k>n}C_k$ stand for the product of measure spaces. Define an embedding $X_n\to X_{n+1}$ by setting
$$(f_n,c_{n+1},c_{n+2},\dots)\mapsto(f_n+c_{n+1},c_{n+2},\dots).$$
It is easy to see that this embedding is measure preserving. Then $X_0\subset X_1\subset\cdots$. Let $X:=\bigcup_{n=0}^\infty X_n$ denote the inductive limit of the sequence of measure spaces X_n and let μ denote the corresponding measure on X. Then μ is σ-finite. It is infinite if and only if

$$\lim_{n\to\infty}\frac{h_n^d}{\#C_1\cdots\#C_n}=\infty. \tag{1-3}$$

Given $g \in \mathbb{R}^d$ and $n \in \mathbb{N}$, we set

$$L_g^{(n)} := (F_n \cap (F_n - g)) \times \prod_{k>n} C_k \quad \text{and} \quad R_g^{(n)} := (F_n \cap (F_n + g)) \times \prod_{k>n} C_k.$$

Clearly, $L_g^{(n)} \subset L_g^{(n+1)}$ and $R_g^{(n)} \subset R_g^{(n+1)}$. Define a map $T_g^{(n)}: L_g^{(n)} \to R_g^{(n)}$ by setting

$$T_g^{(n)}(f_n, c_{n+1}, \dots) := (f_n + g, c_{n+1}, \dots).$$

Put

$$L_g := \bigcup_{n=1}^{\infty} L_g^{(n)} \subset X \quad \text{and} \quad R_g := \bigcup_{n=1}^{\infty} R_g^{(n)} \subset X.$$

Then a Borel one-to-one map $T_g: L_g \to R_g$ is well defined by $T_g \restriction L_g^{(n)} = T_g^{(n)}$. Since $h_n \to \infty$, it follows that $\mu(X \setminus L_g) = \mu(X \setminus R_g) = X$ for each $g \in \mathbb{R}^d$. It is easy to verify that $T := (T_g)_{g \in \mathbb{R}^d}$ is a Borel μ-preserving action of $\mathbb{R}^d$.

Definition 1.2. T is called the (C,F)-*action of* $\mathbb{R}^d$ *associated with* $(C_{n+1}, F_n)_{n \geq 0}$.

Each (C,F)-action is of rank one.

Given a Borel subset $A \subset F_n$, we set $[A]_n := \{ x = (x_i)_{i=n}^{\infty} \in X_n \mid x_n \in A \}$ and call it an n-*cylinder* in X. Clearly,

$$[A]_n = \bigsqcup_{c \in C_{n+1}} [A + c]_{n+1}.$$

Notice also that

$$T_g[A]_n = [A + g]_n \text{ for all } g \in \mathbb{R}^d \text{ and } A \subset F_n \cap (F_n - g), \ n \in \mathbb{N}. \tag{1-5}$$

The sequence of all n-cylinders approximates the entire Borel σ-algebra on X when $n \to \infty$.

We state without proof the following standard lemma (see, e.g., Lemma 2.4 from [Da1]).

Lemma 1.3. *Let $\mathcal{P}_n$ be a finite partition of F_n into parallelepipeds such that for each atom Δ of $\mathcal{P}_n$ and an element $c \in C_{n+1}$, the parallelepiped $\Delta + c$ is $\mathcal{P}_{n+1}$-measurable and the maximal diameter of the atoms in $\mathcal{P}_n$ goes to zero. Let S be a measure preserving transformation of X. Then the following holds.*

(i) *The sequence of collections of n-cylinders $\{[A]_n \mid A \subset F_n$ is $\mathcal{P}_n$-measurable$\}$ approximates the entire σ-algebra $\mathfrak{B}$ as $n \to \infty$.*

(ii) *If for each pair of atoms $\Delta_1, \Delta_2 \in \mathcal{P}_n$, there are a subset $A \subset [\Delta_1]_n$ and a μ-preserving one-to-one map $\gamma : A \to [\Delta_2]_n$ such that $\mu(A) > 0.5\mu([\Delta_1]_n)$, and $\gamma x \in \{S^i x \mid i \in \mathbb{Z}\}$ for all $x \in A$ then S is ergodic.*

We will also use the following property of the (C,F)-actions. If T is associated with $(C_{n+1}, F_n)_{n \geq 0}$ then for each $p > 1$, the product action

$$(T_{t_1} \times \cdots \times T_{t_p})_{(t_1, \dots, t_p) \in (\mathbb{R}^d)^p}$$

is the (C,F)-action of $(\mathbb{R}^d)^p$ associated with $(C_{n+1}^p, F_n^p)_{n \geq 0}$. The upper index p means the p-th Cartesian power.

2. Two auxiliary facts

Given a σ-finite measure space (X,μ), we denote by $\mathrm{Aut}(X,\mu)$ the group of all μ-preserving (invertible) transformations of X. It is a Polish group when endowed with the *weak topology* [Aa]. Recall that the weak topology is the weakest topology in which the maps

$$\mathrm{Aut}(X,\mu) \ni T \mapsto \mu(TA\cap B)\in\mathbb{R}$$

are continuous for all subsets $A,B\subset X$ of finite measure.

Given $S\in\mathrm{Aut}(X,\mu)$ and two subsets $A,B\subset X$ with $\mu(A)=\mu(B)<\infty$, we define subsets $A_0,A_1,\dots$ of A as follows:

$$\begin{aligned} &A_0 := A\cap B,\\ &A_i := \Big(A\setminus\bigsqcup_{j=0}^{i-1}A_j\Big)\cap S^{-i}\Big(B\setminus\bigsqcup_{j=0}^{i-1}S^jA_j\Big),\quad i>0. \end{aligned}\tag{2-1}$$

We now let $\mathcal{N}_{S,A,B}:=\min\{i\ge 0\mid \mu(A_0\sqcup\cdots\sqcup A_i)>0.5\mu(A)\}$. If S is ergodic then $A=\bigsqcup_{i\ge 0}A_i$ and hence $\mathcal{N}_{S,A,B}$ is well defined. Denote by $\mathcal{E}$ the subset of all ergodic transformations in $\mathrm{Aut}(X,\mu)$. It is well known that $\mathcal{E}$ is a dense G_δ in $\mathrm{Aut}(X,\mu)$. Since for each $i\ge 0$, the map

$$\mathrm{Aut}(X,\mu)\ni S\mapsto \mu(A_0\sqcup\cdots\sqcup A_i)\in\mathbb{R}$$

is continuous, we obtain the following lemma.

Lemma 2.1. *The map $\mathcal{E}\ni S\mapsto\mathcal{N}_{S,A,B}\in\mathbb{R}$ is upper semicontinuous for all subsets $A,B\subset X$ with $\mu(A)=\mu(B)<\infty$.*

In the case of (C,F)-actions we can say more about the "structure" of the sets A_i, $i=0,\dots,\mathcal{N}_{S,A,B}$. For $q=(q_1,\dots,q_d)\in\mathbb{R}^d$, we let $\|q\|:=\max_{1\le i\le d}|q_i|$.

Lemma 2.2. *Let $(X,\mu,(T_t)_{t\in\mathbb{R}^d})$ be a (C,F)-action of $\mathbb{R}^d$ associated with a sequence $(C_{n+1},F_n)_{n\ge 0}$ such that*

$$a+F_n+C_{n+1}\subset F_{n+1}\tag{2-2}$$

for each $a=(a_1,\dots,a_p)$ with $a_i\ge 0$, $i=1,\dots,p$, and $\|a\|\le 1$. Fix two n-cylinders A and B of equal measure and a transformation $S=T_q$ for some $q\in\mathbb{R}^d_+$. Then the subsets $A_0,A_1,\dots,A_{\mathcal{N}(S,A,B)}$ defined by (2-1) *are $(n+Q\cdot\mathcal{N}(S,A,B))$-cylinders, where Q is any integer greater than $\|q\|$.*

Proof. We let $N:=Q\cdot\mathcal{N}(S,A,B)$. If $A=[\widetilde{A}]_n$ for some $\widetilde{A}\subset F_n$ then $A=[\widehat{A}]_{n+N}$, where $\widehat{A}:=\widetilde{A}+C_{n+1}+\cdots+C_{n+N}\subset F_{n+N}$. From (1-2) and (2-2) we deduce that the sets $\widehat{A}+q,\dots,\widehat{A}+\mathcal{N}(S,A,B)q$ are all contained in F_{n+N}. It remains to use (2-1) and (1-5). $\square$

We also note that $S^iA_i\subset B$ and $S^iA_i\cap S^jA_j=\emptyset$ for all $i,j=0,\dots,\mathcal{N}(S,A,B)$.

3. Proof of the main result

Theorem 3.1. *There exists a (C,F)-flow $T=(T_t)_{t\in\mathbb{R}}$ such that each transformation T_t, $t\neq 0$, has infinite ergodic index.*

Proof. We will construct this flow via an inductive procedure. Fix a sequence of integers $(p_n)_{n\geq 1}$ in which every integer greater then 1 occurs infinitely many times. Suppose that after $n-1$ steps of the construction we have already defined

$$F_0, C_1, F_1, \dots, C_{m_{n-1}}, F_{m_{n-1}}. \tag{3-1}$$

Suppose also that for each $0\leq i\leq m_{n-1}$, a finite partition $\mathcal{P}_i$ of F_i into intervals is chosen in such a way that

— the interval $\Delta+c$ is $\mathcal{P}_{i+1}$-measurable for each atom Δ of $\mathcal{P}_i$, $0\leq i<m_{n-1}$, and each $c\in C_{i+1}$ and

— the length of any atom of $\mathcal{P}_i$ is no more than i^{-1}, $1\leq i\leq m_{n-1}$.

Step n. Consider a rank-one weakly mixing finite measure preserving (C,F)-flow $T^{(n)}=(T_t^{(n)})_{t\in\mathbb{R}}$ associated with a sequence $(C_{k+1,n},F_{k,n})_{k\geq 0}$ such that $F_{0,n}:=F_{m_{n-1}}$. Examples of weakly mixing rank-one finite measure preserving flows are well known—see, e.g., [dJP]. In [DaS2] one can find explicit (C,F)-construction of mixing finite measure preserving flows. Let $(X^{(n)},\mu_n)$ be the space of this action. Since $T^{(n)}$ is weakly mixing, it follows that for each $t>0$, the transformation

$$S_t := T_t^{(n)}\times\cdots\times T_t^{(n)} \ (p_n \text{ times})$$

of the product space $(X^{(n)},\mu_n)^{p_n}$ is ergodic. We note that this space is the space of the (C,F)-action of $\mathbb{R}^{p_n}$ associated with the sequence $(C_{k+1,n}^{p_n},F_{k,n}^{p_n})_{k\geq 0}$ (see our remark at the end of §1). Given $k\geq 0$, let $\mathcal{P}_{k,n}$ be a finite partition of $F_{k,n}$ into intervals such that

— $\mathcal{P}_{0,n}=\mathcal{P}_{m_{n-1}}$,

— the interval $\Delta+c$ is $\mathcal{P}_{k+1,n}$-measurable for each atom Δ of $\mathcal{P}_{k,n}$ and $c\in C_{k+1,n}$ and

— the length of any atom of $\mathcal{P}_{k,n}$ is no more than $(m_{n-1}+k)^{-1}$.

We now let $\mathcal{P}_{k,n}^{p_n}:=\mathcal{P}_{k,n}\times\cdots\times\mathcal{P}_{k,n}$ (p_n times). Then $\mathcal{P}_{k,n}^{p_n}$ is a finite partition of $F_{k,n}^{p_n}$ into parallelepipeds and

— the parallelepiped $\Delta+c$ is $\mathcal{P}_{k+1,n}^{p_n}$-measurable for each atom Δ of $\mathcal{P}_{k,n}^{p_n}$ and $c\in C_{k+1,n}^{p_n}$ and

— the diameter of an atom of $\mathcal{P}_{k,n}^{p_n}$ is no more than $(m_{n-1}+k)^{-p_n}$.

Denote by D_n the maximum of $\mathcal{N}(S_t,[\Delta]_0,[\Delta']_0)$ when Δ and Δ' run independently the atoms of $\mathcal{P}_{0,n}^{p_n}$ and t runs the segment $[n^{-1},n]\subset\mathbb{R}$. It exists by Lemma 2.1. It now follows from Lemma 2.2 that for any pair of parallelepipeds $\Delta,\Delta'\in\mathcal{P}_{0,n}^{p_n}$ and a real $t\in[n^{-1},n]$, there exist nD_n-cylinders $A_1,\dots,A_{D_n}\subset[\Delta]_0$ such that

$$\begin{gathered}\mu_n^{p_n}\left(\bigsqcup_{i=1}^{D_n}A_i\right)>\frac{1}{2}\mu_n^{p_n}([\Delta]_0),\\ S_t^iA_i\subset[\Delta']_0 \text{ for each } 1\leq i\leq D_n \text{ and}\\ S_t^iA_i\cap S_t^jA_j=\emptyset \text{ if } 1\leq i\neq j\leq D_n.\end{gathered}\tag{3-2}$$

We now "continue" the sequence (3-1) by setting

$$C_{m_{n-1}+1} := C_{1,n}, F_{m_{n-1}+1} := F_{1,n}, \dots, C_{m_{n-1}+nD_n} := C_{nD_n,n}.$$

Next, to define $F_{m_{n-1}+nD_n}$ we "double" the set $F_{nD_n,n}$, i.e.

(3-3) $$F_{m_{n-1}+nD_n} := [0, 2a) \text{ if } F_{nD_n,n} = [0, a) \text{ for some } a > 0.$$

It remains to put $m_n := m_{n-1} + nD_n$. The n-th step is now completed.

Continuing this procedure infinitely many times, we obtain the entire sequence $(C_{i+1}, F_i)_{i=0}^{\infty}$. Denote by $T = (T_t)_{t\in\mathbb{R}}$ the associated (C, F)-flow. Let (X, μ) be the space of this flow. It follows from (3-3) that $\lambda_{\mathbb{R}}(F_i) > 2\lambda_{\mathbb{R}}(F_{i-1})\#C_i$ for infinitely many i. Hence $\mu(X) = \infty$. Moreover, a finite partition $\mathcal{P}_i$ of F_i into intervals is fixed such that the conditions of Lemma 1.3 are satisfied. Next, there are one-to-one correspondences (natural identifications) between

— the collection of 0-cylinders in $X^{(n)}$ and the collection of m_{n-1}-cylinders in X and
— the collection of nD_n-cylinders in $X^{(n)}$ and the collection of m_n-cylinders in X.

Moreover, the "dynamics" of $T^{(n)}$ on the nD_n-cylinders is the same as the dynamics of T on the m_n-cylinders. This means the following: if $A, B \subset F^{(n)}_{nD_n}$ and $[B]_{nD_n} = T^{(n)}_w[A]_{nD_n}$ for some $w \in \mathbb{R}$ then $[B]_{m_n} = T_w[A]_{m_n}$. Therefore we deduce from (3-2) that for any pair of parallelepipeds $\Delta, \Delta' \in \mathcal{P}^{p_n}_{m_n-1}$ and a real $t \in [n^{-1}, n]$, there exist m_n-cylinders $A_1, \dots, A_{D_n} \subset [\Delta]_{m_{n-1}}$ such that

$$\mu^{p_n}\left(\bigsqcup_{i=1}^{D_n} A_i\right) > \frac{1}{2}\mu^{p_n}([\Delta]_{m_{n-1}}),$$
$$V_t^i A_i \subset [\Delta']_{m_{n-1}} \text{ for each } 1 \le i \le D_n \text{ and}$$
$$V_t^i A_i \cap V_t^j A_j = \emptyset \text{ if } 1 \le i \ne j \le D_n,$$

where $V_t := T_t \times \cdots \times T_t$ (p_n times). Fix $p > 0$. Passing to a subsequence where $p_n = p$ we now deduce from Lemma 1.3(ii) that $T_t \times \cdots \times T_t$ (p times) is ergodic for each $t > 0$. Hence T_t has infinite ergodic index. $\square$

4. Concluding remarks

4.1. When constructing T, we use only finitely many initial terms of the sequence $(C^{(n)}_{k+1}, F^{(n)}_k)_{k\ge0}$ for each $n > 0$. However to determine "where to stop" (i.e. to determine D_n) we use the weak mixing properties of the auxiliary flow $T^{(n)}$ which depends on the entire *infinite* sequence $(C^{(n)}_{k+1}, F^{(n)}_k)_{k\ge0}$. No upper bound on D_n is found. This means that the construction of T is not *effective*. In this connection we rise a question:

is it possible to find an effective construction for the flow from Theorem 0.1?

We note that the construction in [DaSo] is effective.

4.2. It is possible to strengthen Theorem 0.1 by replacing the infinite ergodic index with a stronger property of power ergodicity. Recall that a measure preserving transformation S is called power ergodic if for each finite sequence $n_1, \dots, n_k$ of nonzero integers the transformation $S^{n_1} \times \cdots \times S^{n_k}$ is ergodic. Only a slight modification of our argument is needed to show the following theorem.

Theorem 4.1. *There exists a rank-one infinite σ-finite measure preserving flow $T = (T_t)_{t\in\mathbb{R}}$ such that the transformation T_t is power weakly mixing for each $t \neq 0$.*

Also, it is easy to extend Theorem 0.1 to actions of $\mathbb{R}^d$.

Theorem 4.2. *For each $d > 1$, there exists a rank-one infinite σ-finite measure preserving action $T = (T_g)_{g\in\mathbb{R}^d}$ of $\mathbb{R}^d$ such that the transformation T_g has infinite ergodic index for each $g \neq 0$.*

We leave the proofs of Theorems 4.1 and 4.2 to the reader.

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